

ON THE BEST APPROXIMATION OF CERTAIN CLASSES OF PERIODIC FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS

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Abstract

We obtain the estimates for the best approximation in the uniform metric of the classes of 2π -periodic functions whose (ψ, β) -derivatives have a given majorant ω of the modulus of continuity. It is shown that the estimates obtained here are asymptotically exact under some natural conditions on the parameters ψ , ω and β defining the classes

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1 Introduction

Let L be the space of 2π -periodic functions summable over the period with the norm $\|f\|_1 = \int_{-\pi}^{\pi} |f(t)| dt$ and let C be the space of 2π -periodic continuous functions f with the norm $\|f\|_C = \max_t |f(t)|$.

Suppose $f \in L$ and

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad (1)$$

is its Fourier series. Suppose also that $\psi(k)$ is an arbitrary numerical sequence and β is a fixed real number ($\beta \in \mathbb{R}$). If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k \cos \left(kx + \frac{\beta\pi}{2} \right) + b_k \sin \left(kx + \frac{\beta\pi}{2} \right) \right)$$

is the Fourier series of a certain function $\varphi \in L$, then φ is called (see, e.g., [8, 9]) the (ψ, β) -derivative of the function f and is denoted by f_{β}^{ψ} . The set of continuous functions $f(x)$ having (ψ, β) -derivatives such that $f_{\beta}^{\psi} \in H_{\omega}$, where

$$H_{\omega} = \{\varphi \in C : |\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|) \quad \forall t', t'' \in \mathbb{R}\},$$

and $\omega(t)$ is a fixed modulus of continuity is usually denoted by $C_{\beta}^{\psi} H_{\omega}$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C_{\beta}^{\psi} H_{\omega}$ become the well-known Weyl-Nagy classes $W_{\beta}^r H_{\omega}$ which, in turn, for $\beta = r$ coincide with the Weyl classes $W_r^r H_{\omega}$ (see, e.g., [9, Chap. 3, Sec. 4, 6]). For natural numbers r and $\beta = r$ we obtain the classes of periodic functions whose r -th derivatives are in the class H_{ω} .

Let \mathfrak{M} be the set of all continuous functions $\psi(t)$ convex downwards for $t \geq 1$ and satisfying the condition $\lim_{t \rightarrow \infty} \psi(t) = 0$.

If $\psi \in \mathfrak{M}'$, where

$$\mathfrak{M}' := \mathfrak{M}'(\beta) = \{\psi : \psi \in \mathfrak{M} \text{ when } \sin \frac{\beta\pi}{2} = 0 \text{ or}$$

$$\psi \in \mathfrak{M} \text{ and } \int_1^\infty \frac{\psi(t)}{t} dt < \infty \text{ when } \sin \frac{\beta\pi}{2} \neq 0\},$$

then the classes $C_\beta^\psi H_\omega$ coincide with the classes of functions $f(x)$, which are representable by the convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^\pi \varphi(x+t) \Psi_\beta(t) dt, \quad \varphi \in H_\omega^0, \quad x \in \mathbb{R} \quad (2)$$

(see, e.g., [8, p. 31]), where $H_\omega^0 = \{\varphi \in H_\omega : \int_{-\pi}^\pi \varphi(t) dt = 0\}$, and $\Psi_\beta(t)$ is a summable function, whose Fourier series have the form $\sum_{k=1}^\infty \psi(k) \cos(kt + \beta\pi/2)$.

The set \mathfrak{M} is very inhomogeneous in the rate of convergence of functions $\psi(t)$ to zero as $t \rightarrow \infty$. This is why it was suggested in [8, pp. 115, 116] (see also [11, Subsec. 1.3]) to select subsets \mathfrak{M}_0 and \mathfrak{M}_C from \mathfrak{M} as follows:

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} : 0 < \mu(t) \leq K < \infty \quad \forall t \geq 1\},$$

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M} : 0 < K_1 \leq \mu(t) \leq K_2 < \infty \quad \forall t \geq 1\},$$

where $\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t)-t}$, $\eta(t) = \eta(\psi; t) = \psi^{-1}(\psi(t)/2)$, $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$, and K, K_1, K_2 are positive constants (possibly dependent on $\psi(\cdot)$). The function $\mu(\psi; t)$ is called the modulus of half-decay of the function $\psi(t)$. It is obvious that $\mathfrak{M}_C \subset \mathfrak{M}_0$. Typical representatives of the set \mathfrak{M}_C are the functions t^{-r} , $r > 0$, representatives of the set $\mathfrak{M}_0 \setminus \mathfrak{M}_C$ are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 0$. Let $\mathfrak{M}'_0 = \mathfrak{M}' \cap \mathfrak{M}_0$. Natural representatives of the set \mathfrak{M}'_0 are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 1$. It is easy to see that if $\beta = 2l$, $l \in \mathbb{Z}$, the set \mathfrak{M}'_0 coincide with \mathfrak{M}_0 . Moreover, since for all $\psi \in \mathfrak{M}_C$

$$\int_n^\infty \frac{\psi(t)}{t} dt \leq K\psi(n), \quad n \in \mathbb{N}, \quad (3)$$

(see [9, p. 204]) then $\mathfrak{M}_C \subset \mathfrak{M}'_0$. Throughout the paper we denote the positive constants that may be different in different relations by $K, K_i, i = 1, 2$.

Let us denote the best approximation of the classes $C_\beta^\psi H_\omega$ by trigonometric polynomials $t_{n-1}(\cdot)$ of order not more than $n-1$ by $E_n(C_\beta^\psi H_\omega)$, that is

$$E_n(C_\beta^\psi H_\omega) = \sup_{f \in C_\beta^\psi H_\omega} \inf_{t_{n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C. \quad (4)$$

As is shown in [8, p. 330] if $\omega(t)$ is an arbitrary modulus of continuity and $\psi \in \mathfrak{M}_C$, $\beta \in \mathbb{R}$ or $\psi \in \mathfrak{M}'_0$, $\beta = 0$, then the following estimates hold for the quantity $E_n(C_\beta^\psi H_\omega)$:

$$K_1\psi(n)\omega(1/n) \leq E_n(C_\beta^\psi H_\omega) \leq K_2\psi(n)\omega(1/n). \quad (5)$$

When $\psi(k) = k^{-r}$, $r > 0$, $\beta \in \mathbb{R}$, the orders of decrease of quantity (4) have been known earlier [2] (see also [14, p. 508]).

It should be noted that unlike order estimates, exact values for the quantity $E_n(C_\beta^\psi H_\omega)$ have been found for $\psi(k) = k^{-r}$, $r \in \mathbb{N}$, $\beta = r$ and for the convex upwards modulus of continuity by Korneichuk [4] (see also [5, p. 319], [1, p. 344]). The similar problem on the class of real-valued functions defined on the entire real axis and having the r -th continuous derivatives $f^{(r)}$ such that $\omega(f^{(r)}; t) \leq \omega(t)$, $t \in [0, \infty)$, is solved in the paper of Ganzburg [3].

The aim of the present work is to study the rate of decrease of quantity (4) when $\psi \in \mathfrak{M}'_0$ and $\beta \in \mathbb{R}$.

2 Main Results

The following statements are true.

Theorem 1. *Let $\psi \in \mathfrak{M}'_0$, $\beta \in \mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. Then, as $n \rightarrow \infty$,*

$$E_n(C_\beta^\psi H_\omega) = \frac{\theta_n(\omega)}{\pi} \left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n), \quad (6)$$

where $\theta_n(\omega) \in [2/3, 1]$ and $O(1)$ is a quantity uniformly bounded in n and β . If $\omega(t)$ is a convex upwards modulus of continuity, then $\theta_n(\omega) = 1$.

We give an example of functions ψ and ω for which (6) is an asymptotic formula.

Example 1. Let $\psi(t) = \ln^{-\gamma}(t+1)$, $\gamma > 1$, $\beta \neq 2l$, $l \in \mathbb{Z}$ and

$$\omega(t) = \begin{cases} 0, & t = 0, \\ \ln^{-\alpha}\left(\frac{1}{t} + 1\right), & t > 0, \quad 0 < \alpha \leq 1. \end{cases}$$

Then by virtue of (6) the following asymptotic formula holds as $n \rightarrow \infty$:

$$E_n(C_\beta^\psi H_\omega) = \ln^{-(\gamma+\alpha)}(n+1) \left(\frac{1}{\pi(\gamma+\alpha-1)} \left| \sin \frac{\beta\pi}{2} \right| \ln n + O(1) \right),$$

where $O(1)$ is a quantity uniformly bounded in n and β .

Note that if

$$\lim_{n \rightarrow \infty} \frac{|\psi'(n)|n}{\psi(n)} = 0, \quad \psi'(n) := \psi'(n+), \quad (7)$$

and

$$\lim_{n \rightarrow \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0, \quad \omega'(1/n) := \omega'(1/n+), \quad (8)$$

then equalities

$$\lim_{n \rightarrow \infty} \frac{\psi(n)\omega(1/n)}{\int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt} = \lim_{n \rightarrow \infty} \frac{|\psi'(n)|n}{\psi(n)} + \lim_{n \rightarrow \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0,$$

are valid.

Therefore from Theorem 1 we obtain.

Corollary 1. Assume that $\psi \in \mathfrak{M}'_0$, $\beta \neq 2l$, $l \in \mathbb{Z}$, $\omega(t)$ is a convex upwards modulus of continuity and conditions (7) and (8) are fulfilled. Then the following asymptotic formula holds as $n \rightarrow \infty$:

$$E_n(C_\beta^\psi H_\omega) = \frac{1}{\pi} \left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n),$$

where $O(1)$ is a quantity uniformly bounded in n and β .

The functions ψ and ω from Example 1 can serve as an example of the functions which satisfy conditions (7) and (8), respectively.

Relation (6) implies that if $\psi \in \mathfrak{M}'_0$ and

$$\left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \frac{\omega(t)}{t} dt = O(1)\omega(1/n), \quad \beta \in \mathbb{R}, \quad (9)$$

or $\psi \in \mathfrak{M}_C$ (see (3)), then

$$E_n(C_\beta^\psi H_\omega) = O(1)\psi(n)\omega(1/n).$$

Taking into account that function $\psi(t)$ is monotonically decreasing for $t \geq 1$ and using the estimate

$$E_n(C_\beta^\psi H_\omega) \geq K\psi(n)\omega(1/n) \quad \forall \psi \in \mathfrak{M}', \quad (10)$$

(see [8, pp. 329, 330]), by virtue of relation (6) we arrive at the following statement:

Corollary 2. Let $\beta \in \mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. If $\psi \in \mathfrak{M}_C$ or $\psi \in \mathfrak{M}'_0$ and $\omega(t)$ satisfies condition (9), then

$$K_1\psi(n)\omega(1/n) \leq E_n(C_\beta^\psi H_\omega) \leq K_2\psi(n)\omega(1/n), \quad (11)$$

where K_1 and K_2 are positive constants.

Thus, estimates (5) obtained by Stepanets [8, p. 330] (see also [9, Chap. 5, Sec. 22; Chap. 7, Sec. 4]) for the arbitrary modulus of continuity $\omega(t)$ and for $\psi \in \mathfrak{M}_C$, $\beta \in \mathbb{R}$ or for $\psi \in \mathfrak{M}'_0$, $\beta = 0$, hold also in the case when $\psi \in \mathfrak{M}'_0$, $\beta \neq 0$ and $\omega(t)$ satisfies condition (9). For example, the function $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, satisfies (9).

3 Proof of Theorem 1

Suppose that all conditions of the theorem are satisfied. Let us carry out the proof in two stages.

1. We shall find an upper estimate for $E_n(C_\beta^\psi H_\omega)$.

We set

$$U_{n-1}^\psi(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{\psi(n)}{\psi(k)} \frac{k^2}{n^2} \right) (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}, \quad (12)$$

where a_k and b_k are the Fourier coefficients of a function $f \in C_\beta^\psi H_\omega$. Show that for the quantity

$$\mathcal{E}_n(C_\beta^\psi H_\omega) = \sup_{f \in C_\beta^\psi H_\omega} \|f(\cdot) - U_{n-1}^\psi(f; \cdot)\|_C$$

the inequality

$$\mathcal{E}_n(C_\beta^\psi H_\omega) \leq \frac{1}{\pi} \left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n), \quad (13)$$

is true. Since

$$E_n(C_\beta^\psi H_\omega) \leq \mathcal{E}_n(C_\beta^\psi H_\omega), \quad (14)$$

then the required upper estimate for $E_n(C_\beta^\psi H_\omega)$ follows from (13).

For further reasoning, we need the one statement, which follows from the book [8, p. 65]. We will give a few notations before formulating it. Let f be a summable function, whose Fourier series have the form (1). Further, let $\lambda_n = \{\lambda_1(u), \lambda_2(u), \dots, \lambda_n(u)\}$ be a collection of continuous functions on $[0, 1]$ such that $\lambda(k/n) = \lambda_k^{(n)}$, $k = \overline{0, n}$, $n \in \mathbb{N}$, where $\lambda_k^{(n)}$ are elements of the triangular matrix $\Lambda = \|\lambda_k^{(n)}\|$, $k = \overline{1, n}$, $\lambda_0^{(n)} = 1$, that determine a polynomial of the form

$$U_n(f; x; \Lambda) = \frac{a_0}{2} + \sum_{k=1}^n \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}. \quad (15)$$

The following statement is true:

Lemma A [8, p. 65]. Suppose that $f \in C_\beta^\psi H_\omega$ and $\tau_n(u)$ is the continuous function defined by relation

$$\tau_n(u) = \tau_n(u; \lambda; \psi) = \begin{cases} (1 - \lambda_n(u))\psi(1)nu, & 0 \leq u \leq \frac{1}{n}, \\ (1 - \lambda_n(u))\psi(nu), & \frac{1}{n} \leq u \leq 1, \\ \psi(nu), & u \geq 1, \end{cases} \quad (16)$$

and such that its Fourier transform

$$\widehat{\tau}_n(t) := \widehat{\tau}_n(t; \beta) = \frac{1}{\pi} \int_0^\infty \tau_n(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du, \quad \beta \in \mathbb{R},$$

is summable on the whole real line, i.e. $\int_{-\infty}^\infty |\widehat{\tau}_n(t)| dt < \infty$. Then at any point x the following equality holds:

$$f(x) - U_n(f; x; \Lambda) = \int_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{n}\right) \widehat{\tau}_n(t) dt, \quad n \in \mathbb{N}. \quad (17)$$

Using Lemma A, let us show that

$$f(x) - U_{n-1}^\psi(f; x) = \int_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{n}\right) \widehat{\tau}_n(t) dt \quad \forall f \in C_\beta^\psi H_\omega, \quad n \in \mathbb{N}, \quad (18)$$

where $\widehat{\tau}_n(t)$ is the Fourier transform of the function

$$\tau_n(u) = \tau_n(u; \psi) = \begin{cases} \psi(n)u^2, & 0 \leq u \leq 1, \\ \psi(nu), & u \geq 1. \end{cases} \quad (19)$$

Since polynomial (12) can be represented in the form

$$U_{n-1}^\psi(f; x) = \frac{a_0}{2} + \sum_{k=1}^n \lambda^\psi(k/n) (a_k \cos kx + b_k \sin kx),$$

where $\lambda^\psi(k/n)$ are the values of continuous function

$$\lambda^\psi(u) = \lambda_n^\psi(u) = \begin{cases} 1 - \frac{\psi(n)}{\psi(1)} \frac{u}{n}, & 0 \leq u \leq \frac{1}{n}, \\ 1 - \frac{\psi(n)}{\psi(nu)} u^2, & \frac{1}{n} \leq u \leq 1 \end{cases} \quad (20)$$

at the points $u = k/n$ and

$$\tau_n(u) = \tau_n(u; \psi) = \begin{cases} (1 - \lambda^\psi(u))\psi(1)nu, & 0 \leq u \leq \frac{1}{n}, \\ (1 - \lambda^\psi(u))\psi(nu), & \frac{1}{n} \leq u \leq 1, \\ \psi(nu), & u \geq 1, \end{cases}$$

then it follows from Lemma A that for proving (18) it is sufficient to establish the inequality

$$\int_{-\infty}^{\infty} |\hat{\tau}_n(t)| dt < \infty. \quad (21)$$

With this aim we put

$$\mu_n(u) = \begin{cases} \psi(n)(u^2 - u), & 0 \leq u \leq 1, \\ 0, & u \geq 1, \end{cases} \quad \nu_n(u) = \tau_n(u) - \mu_n(u).$$

Integrating twice by parts, we get

$$\hat{\mu}_n(t) := \hat{\mu}_n(t; \beta) = \frac{1}{\pi} \int_0^\infty \mu_n(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = \frac{O(1)}{t^2}, \quad t > 0,$$

which yields

$$\int_{|t| \geq 1} |\hat{\mu}_n(t)| dt < \infty. \quad (22)$$

It is obvious that

$$\int_{|t| \leq 1} |\hat{\mu}_n(t)| dt < \infty. \quad (23)$$

Taking (22), (23) together and using the estimates

$$\int_{-\infty}^{\infty} |\hat{\nu}_n(t)| dt < \infty \quad \forall \psi \in \mathfrak{M}'_0$$

(see, e.g., [9, p. 174]) and

$$|\hat{\tau}_n(t)| \leq |\hat{\mu}_n(t)| + |\hat{\nu}_n(t)|,$$

we obtain (21).

Furthermore, since the function $\tau_n(u)$ satisfies all conditions of Lemma 3 from [13] according to which

$$\tau_n(u) = \int_{-\infty}^{\infty} \cos\left(ut + \frac{\beta\pi}{2}\right) \hat{\tau}_n(t) dt, \quad u \geq 0,$$

we have

$$\int_{-\infty}^{\infty} \hat{\tau}_n(t) dt = \frac{\tau_n(0)}{\cos \frac{\beta\pi}{2}} = 0, \quad \beta \neq 2l - 1, \quad l \in \mathbb{Z}.$$

If $\beta = 2l - 1$, $l \in \mathbb{Z}$, the equality $\int_{-\infty}^{\infty} \hat{\tau}_n(t) dt = 0$ is obvious, because $\hat{\tau}_n(t)$ is odd. Hence, starting from (18) we can write

$$f(x) - U_{n-1}^{\psi}(f; x) = \int_{-\infty}^{\infty} \left(f_{\beta}^{\psi}\left(x + \frac{t}{n}\right) - f_{\beta}^{\psi}(x) \right) \hat{\tau}_n(t) dt \quad \forall f \in C_{\beta}^{\psi} H_{\omega}, \quad n \in \mathbb{N}. \quad (24)$$

Since $f_{\beta}^{\psi} \in H_{\omega}^0$ and, as it is not hard to see, for every $\varphi \in H_{\omega}^0$ function $\varphi_1(u) = \varphi(u + h)$, $h \in \mathbb{R}$, also belongs to H_{ω}^0 , then using the notation

$$\delta(t, \varphi) = \varphi(t) - \varphi(0),$$

it follows from (24) that

$$\begin{aligned} \mathcal{E}_n(C_{\beta}^{\psi} H_{\omega}) &\leq \sup_{\varphi \in H_{\omega}^0} \left| \int_{-\infty}^{\infty} \left(\varphi\left(\frac{t}{n}\right) - \varphi(0) \right) \hat{\tau}_n(t) dt \right| \\ &= \sup_{\varphi \in H_{\omega}^0} \left| \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \hat{\tau}_n(t) dt \right|. \end{aligned} \quad (25)$$

Now we shall simplify the integral in the right-hand side of (25) without loss of its principal value. The following relations are true:

$$\begin{aligned} &\int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \hat{\tau}_n(t) dt \\ &= \frac{\cos \frac{\beta\pi}{2}}{\pi} \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_0^{\infty} \tau_n(u) \cos ut du dt \\ &\quad - \frac{\sin \frac{\beta\pi}{2}}{\pi} \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_0^{\infty} \tau_n(u) \sin ut du dt \\ &= \frac{\cos \frac{\beta\pi}{2}}{\pi} \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_0^{\infty} \tau_n(u) \cos ut du dt \\ &\quad - \frac{\sin \frac{\beta\pi}{2}}{\pi} \left(\int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_0^{\infty} \tau_n(u) \sin ut du dt \right. \\ &\quad \left. + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_0^1 \tau_n(u) \sin ut du dt \right. \\ &\quad \left. + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_1^{\infty} \psi(nu) \sin ut du dt \right). \end{aligned} \quad (26)$$

Integrating by parts, taking into account the equality $\tau_n(0) = \tau_n(\infty) = 0$ and assuming that $\psi'(u) := \psi'(u+)$, we have

$$\begin{aligned} &\int_0^{\infty} \tau_n(u) \cos ut du = -\frac{1}{t} \int_0^{\infty} \tau_n'(u) \sin ut du \\ &= -\frac{2\psi(n)}{t} \int_0^1 u \sin ut du - \frac{n}{t} \int_1^{\infty} \psi'(nu) \sin ut du \end{aligned} \quad (27)$$

and similarly,

$$\int_0^\infty \tau_n(u) \sin ut \, du = \frac{2\psi(n)}{t} \int_0^1 u \cos ut \, du + \frac{n}{t} \int_1^\infty \psi'(nu) \cos ut \, du. \quad (28)$$

Combining (26)–(28), we obtain

$$\begin{aligned} & \int_{-\infty}^\infty \delta\left(\frac{t}{n}, \varphi\right) \widehat{\tau}_n(t) \, dt \\ &= -\frac{\sin \frac{\beta\pi}{2}}{\pi} \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_1^\infty \psi(nu) \sin ut \, du \, dt + r_n(\psi, \varphi, \beta), \quad \varphi \in H_\omega^0, \quad n \in \mathbb{N}, \end{aligned} \quad (29)$$

where

$$\begin{aligned} r_n(\psi, \varphi, \beta) &= \\ &= \frac{\cos \frac{\beta\pi}{2}}{\pi} \left(-2\psi(n) \int_{-\infty}^\infty \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_0^1 u \sin ut \, du \, dt \right. \\ &\quad \left. - n \int_{-\infty}^\infty \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_1^\infty \psi'(nu) \sin ut \, du \, dt \right) \\ &\quad - \frac{\sin \frac{\beta\pi}{2}}{\pi} \left(2\psi(n) \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_0^1 u \cos ut \, du \, dt \right. \\ &\quad \left. + n \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_1^\infty \psi'(nu) \cos ut \, du \, dt \right. \\ &\quad \left. + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_0^1 \tau_n(u) \sin ut \, du \, dt \right) =: \frac{\cos \frac{\beta\pi}{2}}{\pi} \sum_{i=1}^2 J_{i,n} - \frac{\sin \frac{\beta\pi}{2}}{\pi} \sum_{i=3}^5 J_{i,n}. \end{aligned} \quad (30)$$

Let us show that

$$r_n(\psi, \varphi, \beta) = O(1)\psi(n)\omega(1/n). \quad (31)$$

Since for $t \in [-1, 1]$ the quantity

$$\frac{1}{t} \int_0^1 u \sin ut \, du$$

is bounded by a constant, then using the inequality $|\delta(t, \varphi)| \leq \omega(|t|)$, we get

$$J_{1,n} = -2\psi(n) \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_0^1 u \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n). \quad (32)$$

To estimate the integral in (32) we establish the following auxiliary statements.

Lemma 1. *On every interval $(x_k^{(i)}, x_{k+1}^{(i)})$, $x_k^{(i)} = (2k - 1 + i)\pi/2a$, $i = 0, 1$, $k \in \mathbb{N}$, $a > 0$, the function*

$$\int_x^\infty \frac{1}{t} \int_0^a u^s \sin\left(ut + \frac{i\pi}{2}\right) \, du \, dt, \quad x > 0, \quad s \geq 1,$$

has at least one zero.

Proof. We will give a proof of the lemma only for the case $i = 0$, because the proof in case $i = 1$ is similar. On the basis of the estimate $|\int_x^\infty \frac{\sin t}{t} dt| \leq \frac{2}{x}$, $x > 0$ (see, e.g., [12, p. 5], [7, p. 191]) it is simple to see that the integral

$$\int_x^\infty \frac{u^s \sin ut}{t} dt = u^s \int_{ux}^\infty \frac{\sin t}{t} dt$$

converges uniformly with respect to $u \in [0, a]$, $a > 0$. Therefore, changing the order of integration, we obtain

$$S(x) := \int_x^\infty \frac{1}{t} \int_0^a u^s \sin ut du dt = \int_0^a u^s \int_x^\infty \frac{\sin ut}{t} dt du.$$

Making the change of variables and integrating by parts, we have

$$\begin{aligned} S(x) &= \int_0^a u^s \int_{ux}^\infty \frac{\sin t}{t} dt du \\ &= \frac{1}{s+1} \left(a^{s+1} \int_{ax}^\infty \frac{\sin t}{t} dt + \int_0^a u^s \sin ux du \right) \\ &= \frac{1}{s+1} \left(a^{s+1} \int_{ax}^\infty \frac{\sin t}{t} dt - a^s \frac{\cos ax}{x} + \frac{s}{x} \int_0^a u^{s-1} \cos ux du \right). \end{aligned}$$

Hence, taking into account the equation

$$\int_{ax}^\infty \frac{\sin t}{t} dt = \frac{\cos ax}{ax} - \int_{ax}^\infty \frac{\cos t}{t^2} dt$$

we get

$$S(x) = \frac{1}{s+1} \left(-a^{s+1} \int_{ax}^\infty \frac{\cos t}{t^2} dt + \frac{s}{x^{s+1}} \int_0^{ax} u^{s-1} \cos u du \right). \quad (33)$$

On every interval (t_j, t_{j+1}) , $t_j = (2j+1)\pi/2$, $j = 0, 1, \dots$, the function $\int_x^\infty \frac{\cos t}{t^2} dt$ vanishes with a change of sign at some point \tilde{x}_j . Since

$$\int_{\pi/2}^\infty \frac{\cos t}{t^2} dt = - \int_{\pi/2}^\infty \frac{\sin t}{t} dt < 0,$$

then for any $k \in \mathbb{N}$

$$\text{sign} \int_{(2k-1)\pi/2}^\infty \frac{\cos t}{t^2} dt = (-1)^k. \quad (34)$$

Further, we have

$$\int_0^{(2k-1)\pi/2} u^{s-1} \cos u du = \alpha_0 + \sum_{j=1}^{k-1} \alpha_j,$$

where

$$\alpha_0 = \int_0^{\pi/2} u^{s-1} \cos u du, \quad \alpha_j = \int_{(2j-1)\pi/2}^{(2j+1)\pi/2} u^{s-1} \cos u du.$$

If $k = 1$, then

$$\text{sign} \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \text{sign} \alpha_0 = 1. \quad (35)$$

Let $k = 2, 3, \dots$. Since the function u^{s-1} does not decrease ($s \geq 1$) for $u \geq 0$, we can write

$$|\alpha_0| < |\alpha_j| \leq |\alpha_{j+1}|, \quad j \geq 1,$$

and respectively

$$\begin{aligned} & \text{sign} \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du \\ &= \text{sign} \int_{(2k-3)\pi/2}^{(2k-1)\pi/2} u^{s-1} \cos u \, du = (-1)^{k+1}, \quad k = 2, 3, \dots \end{aligned} \quad (36)$$

Taking account of (33)–(36), we have

$$\text{sign} S\left(\frac{2k-1}{2a}\pi\right) = (-1)^{k+1}, \quad k \in \mathbb{N}, \quad a > 0. \quad (37)$$

The function $S(x)$ is continuous for any $x > 0$. Therefore, it follows from (37) that on every interval (x_k, x_{k+1}) , where $x_k = (2k-1)\pi/2a$, $k \in \mathbb{N}$, $a > 0$, the function $S(x)$ has the required zero. Lemma 1 is proved. \blacksquare

Lemma 2. Let $\varphi \in H_\omega$, $1 \leq a \leq n$, $n \in \mathbb{N}$ and $s \geq 1$. Then for $i = 0, 1$, the following estimate holds:

$$\int_{|t| \geq 1} \left(\varphi\left(\frac{t}{n}\right) - \varphi(0) \right) \frac{1}{t} \int_0^{a/n} u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt = O(1)\omega(1/n), \quad (38)$$

where $O(1)$ is a quantity uniformly bounded in n , φ , a and s .

Proof. Making the change of variables, we get

$$\begin{aligned} & \int_{|t| \geq 1} \left(\varphi\left(\frac{t}{n}\right) - \varphi(0) \right) \frac{1}{t} \int_0^{a/n} u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt \\ &= \frac{1}{n^{s+1}} \int_{|t| \geq 1/n} (\varphi(t) - \varphi(0)) \frac{1}{t} \int_0^a u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt, \quad i = 0, 1. \end{aligned} \quad (39)$$

Let us denote by $t_k^{(i)}$ the zero of function

$$\int_x^\infty \frac{1}{t} \int_0^a u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt, \quad i = 0, 1,$$

on interval $(x_k^{(i)}, x_{k+1}^{(i)})$, $x_k^{(i)} = \frac{2k-1+i}{2a}\pi$, which exists according to Lemma 1. Using the notation $\delta(t) = \varphi(t) - \varphi(0)$, we have

$$\begin{aligned} & \left| \int_{1/n}^\infty \delta(t) \frac{1}{t} \int_0^a u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt \right| \\ &= \left| \int_{1/n}^{t_1^{(i)}} \delta(t) \frac{1}{t} \int_0^a u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt \right| \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} (\delta(t) - \delta(t_k^{(i)})) \frac{1}{t} \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du dt \Big| \\
& \leq \omega(t_1^{(i)}) \int_{1/n}^{t_1^{(i)}} \frac{1}{t} \left| \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du \right| dt \\
& \quad + \omega(\Delta_i) \int_{t_1^{(i)}}^{\infty} \frac{1}{t} \left| \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du \right| dt, \tag{40}
\end{aligned}$$

where $\Delta_i = \sup_k (t_{k+1}^{(i)} - t_k^{(i)})$. Since $t_1^{(i)} < \frac{2\pi}{a}$ and $\Delta_i < \frac{2\pi}{a}$, it follows from (40) that

$$\begin{aligned}
& \left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du dt \right| \\
& < \omega \left(\frac{2\pi}{a} \right) \int_{1/n}^{\infty} \frac{1}{t} \left| \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du \right| dt. \tag{41}
\end{aligned}$$

After integrating by parts it is easy to see, that

$$\left| \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du \right| \leq \frac{2a^s}{t}, \quad t > 0, \quad i = 0, 1. \tag{42}$$

From (41) and (42) follows the inequality

$$\begin{aligned}
& \left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du dt \right| < 2a^s \omega \left(\frac{2\pi}{a} \right) \int_{1/n}^{\infty} \frac{dt}{t^2} = 2a^s \omega \left(\frac{2\pi}{a} \right) n \\
& \leq 2a^s \left(\frac{2\pi n}{a} + 1 \right) \omega \left(\frac{1}{n} \right) n < 8a^{s-1} \pi n^2 \omega \left(\frac{1}{n} \right) \leq 8\pi n^{s+1} \omega \left(\frac{1}{n} \right), \quad i = 0, 1. \tag{43}
\end{aligned}$$

The estimate

$$\int_{-\infty}^{-1/n} \delta(t) \frac{1}{t} \int_0^a u^s \sin \left(ut + \frac{i\pi}{2} \right) du dt = O(1) n^{s+1} \omega(1/n), \quad i = 0, 1 \tag{44}$$

is similarly proved. Comparing relations (43), (44) and (39), we obtain (38). Lemma 2 is proved. \blacksquare

Applying Lemma 2 to the integral in (32) and, at the same time, to $J_{3,n}$, we have

$$J_{1,n} = O(1) \psi(n) \omega(1/n), \tag{45}$$

$$J_{3,n} = O(1) \psi(n) \omega(1/n). \tag{46}$$

In the monograph [9, pp. 212, 216, see relations (4.26') and (4.42), (4.45), (4.46)] it is shown, that

$$J_{2,n} = O(1) \psi(n) \omega(1/n) \quad \forall \psi \in \mathfrak{M}_0 \tag{47}$$

and

$$J_{4,n} = O(1) \psi(n) \omega(1/n) \quad \forall \psi \in \mathfrak{M}'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z}. \tag{48}$$

Since $|\tau_n(u)| \leq \psi(n)$, $u \in [0, 1]$, it is clear that

$$J_{5,n} = O(1)\psi(n)\omega(1/n). \quad (49)$$

Comparing (30), (45)–(49), we arrive at (31). Then from (29) for any function $\varphi \in H_\omega^0$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \widehat{\tau}_n(t) dt \\ &= -\frac{\sin \frac{\beta\pi}{2}}{\pi} \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_1^\infty \psi(nu) \sin ut du dt + O(1)\psi(n)\omega(1/n) \\ &= -\frac{\sin \frac{\beta\pi}{2}}{\pi} \int_0^1 \left(\delta\left(\frac{t}{n}, \varphi\right) - \delta\left(-\frac{t}{n}, \varphi\right) \right) \int_1^\infty \psi(nu) \sin ut du dt \\ &\quad + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}'_0, \quad \beta \in \mathbb{R}. \end{aligned} \quad (50)$$

Since

$$\int_1^\infty \psi(nu) \sin ut du > 0, \quad t \in (0, 1], \quad \psi \in \mathfrak{M}', \quad \beta \neq 2l, \quad l \in \mathbb{Z} \quad (51)$$

(see, e.g., [10, p. 143]) and

$$\begin{aligned} & \int_0^1 \omega\left(\frac{2t}{n}\right) \int_1^\infty \psi(nu) \sin ut du dt \\ &= \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z}, \end{aligned} \quad (52)$$

(see [6, p. 632]), from (25) and (50) we obtain (13). Putting together inequalities (13) and (14) we find a required estimate for quantity (4):

$$\begin{aligned} E_n(C_\beta^\psi H_\omega) &\leq \frac{1}{\pi} \left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt \\ &\quad + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}'_0, \quad \beta \in \mathbb{R}. \end{aligned} \quad (53)$$

2. Let us find a lower bound for $E_n(C_\beta^\psi H_\omega)$.

Let $\varphi_n(t)$ be an odd $2\pi/n$ -periodic function defined on $[0, \pi/n]$ by the equalities

$$\varphi_n(t) = \begin{cases} \frac{c_\omega}{2} \omega(2t), & t \in [0, \pi/2n], \\ \frac{c_\omega}{2} \omega\left(\frac{2\pi}{n} - 2t\right), & t \in [\pi/2n, \pi/n], \end{cases}$$

where $c_\omega = 1$ if $\omega(t)$ is a convex upwards modulus of continuity and $c_\omega = 2/3$ otherwise. As shown in [8, pp. 83–85] if $\omega(t)$ is an arbitrary modulus of continuity, then

$$|\varphi_n(t') - \varphi_n(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in [-\pi/2n, \pi/2n].$$

This implies that

$$|\varphi_n(t') - \varphi_n(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in \mathbb{R},$$

and, hence, $\varphi_n \in H_\omega$. We denote by $f^*(\cdot)$ the function from the set $C_\beta^\psi H_\omega$, $\psi \in \mathfrak{M}'$, whose (ψ, β) -derivative $f_\beta^{*\psi}(t)$ coincides with the function $\varphi_n(t)$ on a period. By relations (2), such a function $f^*(\cdot)$ exists.

In virtue of formula (3.4) from the book [8, Chap. 2, Subsec. 3.1] the following equality holds for any $f \in C_\beta^\psi H_\omega$, $\psi \in \mathfrak{M}'$:

$$\begin{aligned} & f(x) - U_{n-1}(f; x; \Lambda) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_\beta^\psi(x+t) \left(\sum_{k=1}^{\infty} \psi(k) \cos \left(kt + \frac{\beta\pi}{2} \right) \right. \\ & \quad \left. - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left(kt + \frac{\beta\pi}{2} \right) \right) dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \end{aligned} \quad (54)$$

where $U_{n-1}(f; x; \Lambda)$ is a trigonometric polynomial of the form (15), such that $\lambda_n^{(n)} = 0$. Since function $\varphi_n(t)$ is odd $2\pi/n$ -periodic, the equalities

$$\int_{-\pi}^{\pi} \varphi_n(t) \sin kt \, dt = 0, \quad k = 1, 2, \dots, n-1, \quad n \geq 2 \quad (55)$$

(see, e.g., [5, p. 159]) and

$$\varphi_n\left(\frac{i\pi}{n} + t\right) = (-1)^i \varphi_n(t), \quad i \in \mathbb{Z},$$

hold. Then, using relation (54) for $f^*(\cdot)$, we obtain

$$\begin{aligned} & f^*\left(\frac{i\pi}{n}\right) - U_{n-1}(f^*; \frac{i\pi}{n}; \Lambda) \\ &= \frac{(-1)^i}{\pi} \int_{-\pi}^{\pi} \varphi_n(t) \left(\sum_{k=1}^{\infty} \psi(k) \cos \left(kt + \frac{\beta\pi}{2} \right) - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left(kt + \frac{\beta\pi}{2} \right) \right) dt \\ &= \frac{(-1)^i}{\pi} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=1}^{\infty} \psi(k) \cos \left(kt + \frac{\beta\pi}{2} \right) dt \\ &= \frac{(-1)^i}{\pi} \sin \frac{\beta\pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt \, dt, \quad i \in \mathbb{Z}, \quad n = 2, 3, \dots \end{aligned} \quad (56)$$

It is obvious from this that there exist $2n$ points $t_i = \frac{i\pi}{n}$, $i = 0, 1, \dots, 2n-1$, on the period $[0, 2\pi)$ at which the difference

$$f^*(x) - U_{n-1}(f^*; x; \Lambda)$$

takes values with alternating signs. Then by the de la Vallée Poussin theorem [15] (see also [8, p. 312], [9, p. 491]), we find

$$E_n(f^*) \geq \frac{1}{\pi} \left| \sin \frac{\beta\pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt \, dt \right|, \quad \psi \in \mathfrak{M}', \quad (57)$$

where

$$E_n(f^*) = \inf_{t_{n-1}} \|f^*(\cdot) - t_{n-1}(\cdot)\|_C, \quad n \in \mathbb{N}.$$

From (56) and (57) it follows, in particular, that

$$E_n(f^*) \geq |f^*(0) - U_{n-1}(f^*; 0; \Lambda)|, \quad n = 2, 3, \dots \quad (58)$$

Inequality (58) is satisfied for triangular matrix $\Lambda = \|\lambda_k^{(n)}\|$, $k = \overline{1, n}$, such that $\lambda_n^{(n)} = 0$. Let's define its remaining elements in the following way:

$$\lambda_k^{(n)} = \lambda^\psi(k/n), \quad k = \overline{1, n-1}, \quad n \in \mathbb{N},$$

where $\lambda^\psi(\cdot)$ is defined by (20). Since in this case

$$U_{n-1}(f^*; 0; \Lambda) = U_{n-1}^\psi(f^*; 0),$$

then from (58) we obtain, taking the inequality $E_n(C_\beta^\psi H_\omega) \geq E_n(f^*)$ into account,

$$E_n(C_\beta^\psi H_\omega) \geq |f^*(0) - U_{n-1}^\psi(f^*; 0)|, \quad n = 2, 3, \dots, \quad \psi \in \mathfrak{M}'. \quad (59)$$

By virtue of (24) and (50)

$$\begin{aligned} f^*(0) - U_{n-1}^\psi(f^*; 0) &= \int_{-\infty}^{\infty} \left(f_\beta^{*\psi}\left(\frac{t}{n}\right) - f_\beta^{*\psi}(0) \right) \widehat{\tau}_n(t) dt \\ &= -\frac{\sin \frac{\beta\pi}{2}}{\pi} \int_0^1 \left(\varphi_n\left(\frac{t}{n}\right) - \varphi_n\left(-\frac{t}{n}\right) \right) \int_1^\infty \psi(nu) \sin ut du dt \\ &\quad + O(1)\psi(n)\omega(1/n) \\ &= -c_\omega \frac{\sin \frac{\beta\pi}{2}}{\pi} \int_0^1 \omega\left(\frac{2t}{n}\right) \int_1^\infty \psi(nu) \sin ut du dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}'_0. \end{aligned} \quad (60)$$

Combining (51), (52), (59) and (60), we arrive at the desired estimate

$$\begin{aligned} E_n(C_\beta^\psi H_\omega) &\geq \frac{c_\omega}{\pi} \left| \sin \frac{\beta\pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt \\ &\quad + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathfrak{M}'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z}. \end{aligned} \quad (61)$$

From (53) and (61) we obtain formula (6). Theorem 1 is proved. \blacksquare

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